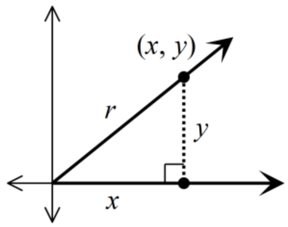
The Six Circular Functions

The six trigonometric functions are the sine (sin), cosine (cos), tangent (tan), cosecant (csc), secant (sec), and cotangent (cot) functions. There are several ways to define these functions of trigonometry.

All of these ratios can be written in terms of an angle in the coordinate plane.

If (x, y) is any point other than the origin on the terminal side of an angle α in standard position and $r = \sqrt{x^2 + y^2}$, then



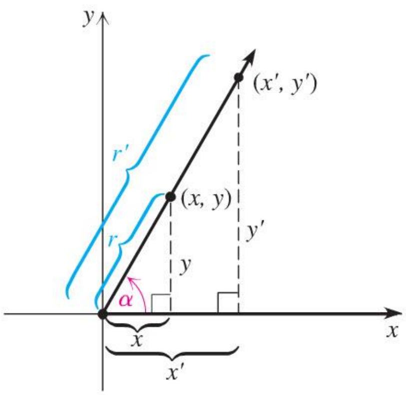
$$\sin\alpha = \frac{\text{opp}}{\text{hyp}} = \frac{y}{r} \qquad \cos\alpha = \frac{\text{adj}}{\text{hyp}} = \frac{x}{r} \qquad \tan\alpha = \frac{\text{opp}}{\text{adj}} = \frac{y}{x}$$

$$\csc\alpha = \frac{\text{hyp}}{\text{opp}} = \frac{r}{y} \qquad \sec\alpha = \frac{\text{hyp}}{\text{adj}} = \frac{r}{x} \qquad \cot\alpha = \frac{\text{adj}}{\text{opp}} = \frac{x}{y}$$

Reciprocal Identities: $\csc\alpha = \dfrac{1}{\sin\alpha}$ $\qquad \sec\alpha = \dfrac{1}{\cos\alpha}$ $\qquad \cot\alpha = \dfrac{1}{\tan\alpha}$

Examples:
Find the values of the six trigonometric functions of the angle α in standard position whose terminal side passes through (-2, -4).

To find the values of the trigonometric functions of the "special angles" on the unit circle (multiples of 30° and 45°). We could choose any point on the terminal side of each angle and the same ratios would result because the triangles used to calculate the ratios are similar.



$$\sin\alpha = \frac{y}{r} = \frac{y'}{r'} \qquad \cos\alpha = \frac{x}{r} = \frac{x'}{r'} \qquad \tan\alpha = \frac{y}{x} = \frac{y'}{x'}$$

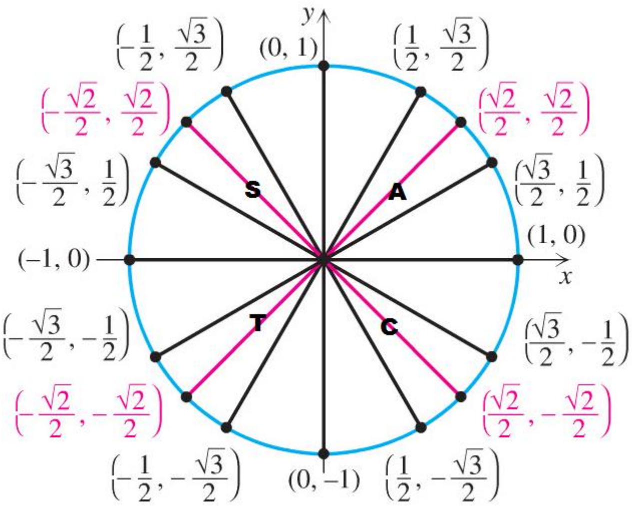
Since $r = 1$ for any point on the unit circle, points on the unit circle are convenient to use for calculating trigonometry ratios.

On the Unit Circle:

$$\sin\alpha = y \qquad \cos\alpha = x \qquad \tan\alpha = \frac{y}{x}$$

$$\csc\alpha = \frac{1}{y} \qquad \sec\alpha = \frac{1}{x} \qquad \cot\alpha = \frac{x}{y}$$

Remember from Chapter 2.2, a good mnemonic to remember which functions are positive in each quadrant is "**A**ll **S**tudents **T**ake **C**alculus":



Tangent and Cotangent Values of Common Angles

θ degrees	θ radians	$\cos(\theta)$	$\sin(\theta)$	$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$	$\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$
0°	0	1	0	0	undefined
30°	$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$	$\sqrt{3}$
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	1
60°	$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\sqrt{3}$	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$
90°	$\frac{\pi}{2}$	0	1	undefined	0

Examples: Find the exact values of the following:

1. $\sin 0°$
2. $\cos \pi$
3. $\tan(-\pi/2)$
4. $\csc(-270°)$
5. $\sin(\pi/4)$
6. $\cos(-225°)$
7. $\cot(5\pi/4)$
8. $\sec 315°$
9. $\sin 30°$
10. $\cos(7\pi/6)$
11. $\tan(-\pi/3)$
12. $\csc 150°$
13. $\cot(3)$
14. $\sec(10)$
15. $\cos(\theta)$, where $\csc(\theta) = -\sqrt{5}$ and θ is a Quadrant IV angle.
16. $\sin(\theta)$, where $\tan(\theta) = 3$ and $\pi < \theta < \frac{3\pi}{2}$.

Finding Angles that satisfy Circular Function Equations

A solution to the equation $\sin\alpha = \frac{1}{2}$ is an angle whose sine is $\frac{1}{2}$. Because $\sin 30° = \frac{1}{2}$ and $\sin 150° = \frac{1}{2}$, θ could be 30° or 150°. Since any angle with the same terminal side as 30° or 150° is also a solution, there are infinitely many solutions.

Examples:

Find all of the angles which satisfy the given equation.

a. $\cos(\theta) = \dfrac{1}{2}$

b. $\sin(\theta) = -\dfrac{1}{2}$

c. $\cos(\theta) = 0$

d. $\sin\theta = \sqrt{3}/2$

e. $\cos\theta = 1$

f. $\tan\theta = 1$

g. $\sec(\theta) = 2$

h. $\tan(\theta) = \sqrt{3}$

i. $\cot(\theta) = -1$

Suppose we are asked to solve an equation such as $\sin(t) = -\dfrac{1}{2}$. As we have already mentioned, the distinction between t as a real number and as an angle $\theta = t$ radians is often blurred. Indeed, we solve $\sin(t) = -\dfrac{1}{2}$ in the exact same manner as we did in the previous examples. Our solution is only cosmetically different in that the variable used is t rather than θ: $t = \dfrac{7\pi}{6} + 2\pi k$ or $t = \dfrac{11\pi}{6} + 2\pi k$ for integers k.